arXiv:hep-lat/9709084v1 22 Sep 1997

hep-lat/9709084

LSUHE No. 266-1997
September, 1997

Ehrenfest theorems for field strength and electric current in Abelian projected SU(2) lattice gauge theory

Giuseppe DiCecio*, Alistair Hart and Richard W. Haymaker†

Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803, USA

We derive an Ehrenfest theorem for SU(2) lattice gauge theory which, after Abelian projection, relates the Abelian field strength and a dynamical electric current and defines these operators for finite lattice spacing. Preliminary results from the ongoing numerical test of the relation are presented, including the contributions from gauge fixing and the Faddeev–Popov determinant (the ghost fields) in the maximally Abelian gauge.

After applying Abelian projection to non–Abelian gauge theories, the result is mathematically identical to a set of charged fields (vector–like in the maximally Abelian (MA) gauge) coupled to an electromagnetic field, governed by a complicated U(1) gauge invariant action. The charged fields have traditionally been ignored, but numerical work suggests that we should reexamine their rôle.

We are presently completing a numerical check of an exact relation for SU(2) lattice averages after Abelian projection to general gauges,

$$\langle \Delta_\mu F_{\mu\nu} - J_\nu \rangle_{\text{static Abelian source}} = 0. \tag{1}$$

J_ν, the electric current in the remaining U(1), contains terms from the external source, the "off-diagonal" gauge potentials, the gauge fixing condition and the Faddeev–Popov determinant:

$$\begin{aligned} J_\nu &= J_\nu^{\text{(source)}} + J_\nu^{\text{(dynamical)}} \\ &+ J_\nu^{\text{(gauge fixing)}} + J_\nu^{\text{(Faddeev–Popov)}}. \end{aligned}$$

Eqn. (1) resembles the Euler-Lagrange–Maxwell equation, of course, satisfied at the extremum of the action. With a suitable choice of lattice operators, however, lattice averages also satisfy this relation. The corresponding relations for U(1) and for SU(3) without gauge fixing are given in Ref. [1]. The term "Ehrenfest theorem" is taken from the context of quantum mechanics, where a classical equation is exactly satisfied for expectation values, e.g. $\frac{d^2}{dt^2}\langle \boldsymbol{r}\rangle = -\langle \boldsymbol{\nabla} V(\boldsymbol{r})\rangle$.

Motivation I: Eqn. (1) defines electric current density on the lattice.

- Unlike pure U(1), an electric current density occurs in the Abelian projected theory, and is capable of screening sources and affecting the string tension; Bali et. al. [2] found that the string tension after Abelian projection to the MA gauge is only 92% of the full SU(2) result.

- The contributions from gauge fixing and the Faddeev–Popov determinant contribution [3] to the current can be measured by making use of this relation. The latter is the contribution from the ghost fields.

- One uses the Ginzburg–Landau theory interpreted as a dual effective theory to model the simulations. One needs to modify this model, however, to include the effects of a dynamical charge density.

Motivation II: Eqn. (1) defines Abelian field strength on the lattice.

- For regions where $J_\nu = 0$, this defines exactly the conservation of flux. It then gives precise meaning to the intuitive picture of the vacuum squeezing the field lines.

- Crucial to this mechanism for confinement is the connection between the spontaneous breaking of a gauge symmetry as indicated by a non-zero vacuum expectation value of the monopole

*Present address: Dept. of Physics, Univ. of Pisa, Italy.
†Presented by R. Haymaker. Work partially supported by the U. S. Department of Energy.

creation operator and the formation of vortices. Both the monopole operator [4,5] and the vortex operators [6] rely on a definition of electric field strength. Therefore a tightening of these definitions could enhance our understanding of this crucial connection.

- One can compare this definition of Abelian field strength with the lattice implementation [7] of the 't Hooft expression [8] which would lead to an exact Ehrenfest theorem only to leading order in the lattice spacing, a.

U(1) Example

Consider the partition function for the U(1) plaquette action, $S_a = \sum_{n,\mu,\nu}(1-\cos\theta_{\mu\nu}(n))$, including a Wilson loop:

$$\mathcal{Z}_W = \int [d\theta_\mu] e^{i\theta_W} \exp\left(-\beta S_a\right).$$

It is invariant under shifting any link angle, $\theta_\mu(n) \to \theta_\mu(n) + \epsilon$. Using this invariance Zach et. al. [1] derived the relation:

$$\frac{\int [d\theta_\mu] \sin\theta_W \left(\frac{1}{a}\Delta_\mu \frac{\sin\theta_{\mu\nu}}{ea^2}\right) e^{-\beta S_a}}{\int [d\theta_\mu] \cos\theta_W e^{-\beta S_a}} = e\frac{\delta_W}{a^3}$$

where $\delta_W = 0$ unless the shifted link lies on the Wilson loop external source, when $\delta_W = \pm 1$. By identifying $\sin\theta_{\mu\nu}/(ea^2)$ as the field strength we then obtain an Ehrenfest theorem of the form:

$$\left\langle \frac{1}{a}\Delta_\mu F_{\mu\nu} - J_\nu^{(\text{static})} \right\rangle_{\text{source}} = 0.$$

The choice of $\theta_{\mu\nu}/(ea^2)$ as the field strength would not lead to an Ehrenfest theorem for finite lattice spacing.

Generalization to SU(2)

The SU(2) Wilson plaquette action, S, gives

$$\mathcal{Z}_W = \int [dU_\mu] e^{i\theta_W} \exp\left(-\beta S\right),$$

where W now indicates an Abelian Wilson loop. Ignoring gauge invariance for the moment, we exploit the invariance of the measure under a right (or left) multiplication of a link variable by a constant SU(2) matrix, $U_\mu \to U_\mu\left(1 - \frac{i}{2}\epsilon\sigma_3\right)$. The derivative of S with respect to ϵ, which we denote as S_μ, inserts a σ_3 in the six plaquettes contiguous to the shifted link. Similarly the Abelian Wilson loop has a σ_3 insertion if it contains the shifted link. This gives the relation

$$\left\langle \frac{2}{ga^3} S_\nu(U) - J_\nu^{(\text{static})} \right\rangle_{\text{Abelian source}} = 0, \quad (2)$$

This can be cast into the form Eqn. (1) using the parametrization of the link matrix:

$$U_\mu = \begin{pmatrix} \cos(\phi_\mu)e^{i\theta_\mu} & \sin(\phi_\mu)e^{i\chi_\mu} \\ -\sin(\phi_\mu)e^{-i\chi_\mu} & \cos(\phi_\mu)e^{-i\theta_\mu} \end{pmatrix}, \quad (3)$$

where $\phi_\mu \in [0, \frac{\pi}{2})$ and θ_μ, $\chi_\mu \in [-\pi, \pi)$. Separate this into the diagonal, $\mathcal{D}_\mu$, and off-diagonal, $\mathcal{O}_\mu$, parts; $U_\mu = \mathcal{D}_\mu + \mathcal{O}_\mu$. Applying this to Eqn. (2) gives

$$\left\langle \frac{1}{a}\Delta_\mu F_{\mu\nu} - J_\nu^{(\text{dynamical})} - J_\nu^{(\text{static})} \right\rangle = 0, \quad (4)$$

where $\frac{1}{a}\Delta_\mu F_{\mu\nu}$ contains only $\mathcal{D}_\mu$ contributions to the links and $J_\nu^{(\text{dynamical})}$ the rest.

Gauge fixing

Gauge fixing complicates the issue, although the essence of the argument goes through as before. Prior to Abelian projection we gauge fix to satisfy $F_i(U;n) = 0$ for $i = 1,2$. When shifting a link $U_\mu \to U_\mu\left(1 - \frac{i}{2}\epsilon\sigma_3\right)$ we must in general perform a simultaneous gauge transformation, $g(n) = 1 - \frac{i}{2}\boldsymbol{\eta}\cdot\boldsymbol{\sigma}$, where $\eta^i(n) \propto \epsilon$, to avoid leaving the gauge condition. S_μ is gauge invariant, but we obtain extra terms from the Wilson loop source and the Faddeev–Popov determinant when we differentiate with respect to ϵ.

The partition function now reads:

$$\mathcal{Z}_W = \int [dU_\mu]\, e^{i\theta_W}\, \Delta_{FP} \prod_{n,i} \delta(F_i(U;n))\, e^{-\beta S}.$$

We are primarily interested in the MA gauge,

$$\begin{aligned} F_i(U;n) \quad &= \quad \frac{1}{2}\sum_\mu \left\{ \mathrm{Tr}\left(\sigma_i U_\mu^\dagger(n)\sigma_3 U_\mu(n)\right)\right. \\ &+ \quad \left.\mathrm{Tr}\left(\sigma_i U_\mu(n-\hat{\mu})\sigma_3 U_\mu^\dagger(n-\hat{\mu})\right)\right\}, \end{aligned}$$

and integrating out the ghost fields gives

$$\Delta_{FP} = \det\left(\frac{\partial F_i(U;n)}{\partial \eta_j(m)}\right)_{F_i(U;n)=0}.$$

The Ehrenfest theorem is now given by

$$\int [dU_\mu]\, e^{i\theta_W}\ \Delta_{FP}\ \prod_{n,i} \delta(F_i(U;n)) \times \left(-\beta S_\mu + \frac{(\Delta_{FP})_\mu}{\Delta_{FP}} + i(\theta_W)_\mu \right) e^{-\beta S} = 0.$$

which is recast as Eqn. (1).

Status and Conclusions

When shifting the link does not conflict with the gauge condition, e.g. when no gauge condition is imposed, no extra gauge transformation is required and we find Eqn. (4) is satisfied exactly.

Ignoring this conflict in the MA gauge, in a normalization where $\langle J_\nu^{(\mathrm{static})} \rangle = 1$ on the source the sum of the first two terms in Eqn. (4) is 1.128 (5). We have used a Abelian plaquette source ($\beta = 2.3$ on a 12^4 lattice). Hence there is a 13% violation of the identity. On the same lattice, in the numerically simpler gauge that diagonalizes all plaquettes in a particular plane, the violation is -23%. In both cases there is a rapidly decreasing, but non-zero signal for the summed terms that extends away from the source where $\langle J_\nu^{(\mathrm{static})} \rangle = 0$.

The corrective gauge transformation at every site that accompanies the shift of a single link introduces a non-locality. The Wilson loop derivative, $(\theta_W)_\mu$, is increased, and picks up a contribution even when the shifted link is not one of those making up the loop. The magnitude of the gauge transformation falls off exponentially with distance, however, and is a small effect. In other words, the derivative of the source is no longer a delta function of position, but is slightly smeared by the gauge fixing.

Inclusion of this reduces the violation of the MA gauge Ehrenfest identity to 3% on a 6^4 lattice at $\beta = 2.3$ (since this is a lattice identity, finite volume and scaling considerations are irrelevant). On such a lattice, the plaquette gauge relation is improved to -15%.

The calculation of the Faddeev–Popov term is incomplete, but preliminary results indicate that $\langle J_\nu^{(\mathrm{Faddeev–Popov})} \rangle$ is relatively small and of the correct sign and magnitude to cancel the remaining violation and satisfy Eqn. (1) both in the MA and plaquette gauges.

The lattice definition of Abelian field strength that follows from this approach is

$$F_{\mu\nu}(n) = \frac{1}{ga^2} \times \mathrm{Tr}\ \left[\sigma_3 \mathcal{D}_\mu(n) \mathcal{D}_\nu(n+\hat{\mu}) \mathcal{D}_\mu^\dagger(n+\hat{\nu}) \mathcal{D}_\nu^\dagger(n) \right].$$

This differs from the lattice version [7] of the 't Hooft Abelian field strength [8]. Both agree in the continuum limit, but the latter would satisfy the Ehrenfest theorem only to leading order in a.

In terms of the link parametrization, Eqn. (3), the field strength is:

$$F_{\mu\nu}(n) = \frac{1}{ga^2} \cos\theta_{\mu\nu}(n) \times \{ \cos\phi_\mu(n) \cos\phi_\nu(n+\hat{\mu}) \cos\phi_\mu(n+\hat{\nu}) \cos\phi_\nu(n) \}.$$

The factors $\cos\phi_\mu(n) \to 1$ in the continuum limit.

REFERENCES

1. M. Zach, M. Faber, W. Kainz and P. Skala, Phys. Lett. **B 358** (1995) 325; P. Skala, M. Faber and M. Zach, Nucl. Phys. **B 494** (1997) 293.
2. G.S. Bali, V. Bornyakov, M. Müller-Preussker and K. Schilling, Phys. Rev. **D 54** (1996) 2863.
3. We thank M.I. Polikarpov and F. Gubarev for pointing out that we should expect a contribution from the Faddeev–Popov determinant.
4. L. Del Debbio, A. Di Giacomo, G. Paffuti and P. Pieri, Phys. Lett. **B 355** (1995) 255; A. Di Giacomo and G. Paffuti, hep-lat/9707003.
5. M.N. Chernodub, M.I. Polikarpov and A.I. Veselov, Nucl. Phys. B (Proc. Suppl.) **49** (1996) 307.
6. V. Singh, D.A. Browne, and R.W. Haymaker, Nucl. Phys. B (Proc. Suppl.) **30** (1993) 658; Phys. Lett. **B 306** (1993) 115; Y. Matsubara, S. Ejiri, and T. Suzuki, Nucl. Phys. B (Proc. Suppl.) **34** (1994) 176; Y. Peng and R.W. Haymaker, Phys. Rev. **D 52** (1995) 3030.
7. K. Bernstein, G. DiCecio and R.W. Haymaker, Phys. Rev. **D 55** (1997) 6730.
8. G. 't Hooft, Nucl. Phys. **B 79** (1974) 276.